MTH 605: Topology I

Practice Assignment II

1. A space X is *weakly locally connected at* x if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at every point, then X is loclally connected.

2. Describe the components and path components of the following spaces.

   (a) $\mathbb{R}_\ell$

   (b) $\mathbb{R}^\infty$ with product and box topologies

3. Show that every compact subspace of a metric space is closed and bounded. Find a metric space in which the converse does

4. Show that if X is compact Hausdorff under two topologies τ and τ', then either $\tau = \tau'$ or they are

5. Let Y be a compact space.

   (a) Show that $\pi_1 : X \times Y \to X$ is a closed map.

   (b) Let Y be a Hausdorff space, and let $f : X \to Y$. Then f is continuous if and only if the *graph* of f, $G_f = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

6. Show that a connected space having more than one point is uncountable.

7. Let $p : X \to Y$ be a surjective continuous map each of whose fibers is compact. Show that if Y is compact, then X is compact.

8. Let X be a compact Hausdorff space. Let $\mathcal{B}$ be a collection of closed connected subsets that are simply ordered under inclusion. Then show that $\cap_{a \in \mathcal{A}} A$ is connected.

9. Establish the following facts.

   (a) $[0, 1]$ is not compact in $\mathbb{R}_K$.

   (b) $\mathbb{R}_K$ is connected, but not path connected

   (c) $[0, 1]$ is not limit point compact in $\mathbb{R}_\ell$.

   (d) Every subset of $\mathbb{R}$ under the cofinite topology is compact.

   (e) $\mathbb{Q}$ is not locallly compact.

10. A space X is *countably compact* if every countable covering of X has a finite subcovering. Show that in a T_1 space X, countable compactness is equivalent to limit point compactness. [Hint: If not finite sucollection of U_n covers X, then choose $x_n \notin U_1 \cup \ldots \cup U_n$ for each n.]

11. Let (X, d) be a compact metric space. Show that every isometry on X is a homeomorphism.

12. Let G be a toplogical group.

(a) Show that if C is a component of G containing the identity element, then $C \trianglelefteq G$.

(b) If G is locally compact and $H \leq G$, then G/H is locally compact.

13. Show that a homeomorphism of locally compact Hausdorff spaces extends to their one-point compactification.

14. Describe the one-point compactification of the following spaces.

    (a) $\mathbb{R}$

    (b) $\mathbb{Z}_+$

    (c) $\mathbb{R}^n$